Numerical simulation of elastic membrane - fluid interaction in a closed reservoir

Yehuda Arav   Michel Bercovier*

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Abstract

In [5] was introduced the problem of 3D elastic membrane-fluid interaction in a closed reservoir. In this work we solved numerically the problem mentioned using a simpler elastic model in 2D. To solve the transient problem with newtonian-viscous fluid, two preliminary steps were taken. First, solving the steady state problem, this problem is an ordinary differential equation that was solved analytically also. The second step was to solve the the transient problem with an ideal fluid. Since the fluid is ideal it serves only as a volume conservator. When solving the problem with a newtonian-viscous fluid the fluid serves not only as a volume conservator, but also carry forces on the membrane.

Problem definition

Following [5] and reducing the 3D model to a 2D model we get a rectangle whose longer edge is parallel to the x-axis and its length is L meters, the length of the shorter edge is H meter. The upper edge, parallel to the x-axis, is a flexible membrane (Fig 1).

Since the model is 2D, the units measuring the quantity of fluid are area (m^2) rather than volume (m^3).

When filling the box with an incompressible fluid of area A > L*H because the membrane is distended.

* Hebrew University, School of Computer Science and Eng., Safra Campus, Jerusalm, 91904, Israel
yeharav,berco@cs.huji.ac.il

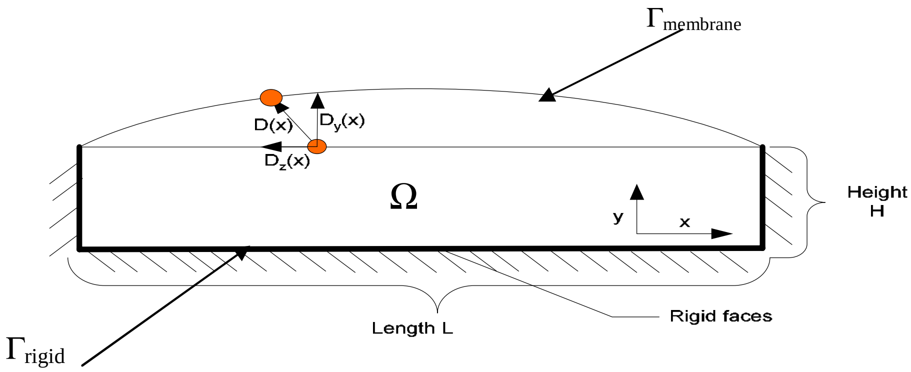

Figure 1 – Framework of the model describing solid-liquid interaction. A box with 3 rigid faces (bottom, left and right) and an elastic membrane (top). D(x) is the displacement, $D_y(x)$ is the displacement at the y-axis, $D_x(x)$ is the displacement at the x-axis, Ω is the computational domain, Γ_{rigid}, and $\Gamma_{membrane}$ are the computational domain boundary.

This work will deal with the following problems:

Problem 1 - Steady-state

1. Given the properties of the membrane and the area of the fluid, find the steady-state shape of the membrane when it is not subject to external forces.

Problem 2 -

2. Given f(x, t) that describes the forces applied to the membrane,
   a. Ideal fluid – find the shape of the flexible face as a function of time under condition of ideal fluid.
   b. Viscous fluid - find the velocity field u(x, t), and the shape of the flexible face as a function of time under condition of viscous fluid.

Solution schemes

Elasticity model

When assuming an isotropic membrane, that is when displacement does not depend on the direction of the load, Navier elastic equations may be applied.

$$\rho_w h \frac{\partial^2 D_y}{\partial t^2} = kGh \frac{\partial^2 D_y}{\partial x^2} + \Phi_1 \tag{1.1}$$

$$\rho_w h \frac{\partial^2 D_x}{\partial t^2} = \frac{Eh}{1-\xi^2} \frac{\partial D_x}{\partial x^2} + \Phi_2 \qquad (1.2)$$

Where D_y and D_x stand for the displacements in the y and x directions respectively, h for the membrane wall thickness, k for the shear correction factor, G for the shear modules, E for the Young modules, ξ for the Poisson ratio (which is equal to ½ for incompressible material), ρ_0 for the muscle wall mass and $\Phi = [\Phi_1\ \Phi_2]^T$ for the force term which is due to external forces (including the stress induced by the fluid).

For simplicity, let us examine the behavior of this model (1) under the condition where the horizontal displacement is much smaller than the vertical displacement. In this case equation (1.2) can be neglected and equation (1.1) reduces to:

$$p_w \frac{\partial^2 \eta}{\partial t^2} + \frac{E}{1-\xi^2} \eta = \frac{\Phi}{h} \qquad (2)$$

where $\Phi = \Phi_1$, and $\eta = D_y(x)$

Equation (2) is an ordinary differential equation of second order and is known as the Independent Rings Model [1, 2].

There is, however, a model of intermediate complexity between (1) and (2) which takes into account the tension of the membrane:

$$\frac{\partial^2 \eta}{\partial t^2} - \beta \frac{\partial^2 \eta}{\partial x^2} + \sigma\eta = f(t) \qquad (3)$$

where $\eta = D_y$ and where D_x is neglected, β and σ are the membrane properties.

When β= 0 the model reduces to the independent rings model (2). This work will use equation (3) as the elasticity model.

Problem 1 – steady-state

Under steady-state, the membrane displacement, $\eta(x,t)$, depends only on the spatial coordinates (and therefore reduces to $\eta(x)$) and on the force imposed on the membrane. At steady-state the problem is defined by:

$$-\beta\frac{d^2\eta}{dx^2}+\sigma\eta-P=f \quad (4)$$

Where equation (4) describes the membrane displacement, $\eta=\eta(x)$; f for the external forces and P for the Lagrange multiplier for constraint (5).

$$\int\eta dx=V_0 \quad (5)$$

where equation (5) accounts for the area conservation, A_0=A-L*H is the initial area. The boundary conditions are η(0) = η(1) = 0.

Since there are no external forces, f(x) is only the internal force, that is, the hydrostatic pressure exerted by fluid on the membrane, P. P is constant or flow would be induced because of pressure difference, changing the membrane displacement which violates the steady-state assumption. Moreover, since the pressure serves as a Lagrange multiplier for equation (5) it must be a scalar.

We will now solve equation (4) analytically and then present a numerical solution. The solution of a second order ordinary linear differential,

$$\frac{d^2y}{dx^2}+a_1\frac{dy}{dx}+a_2y=b(x) \quad (6)$$

At interval I is given by [3],

$$\psi(x)=c_1e^{r_1x}+c_2e^{r_2x}+\frac{1}{r1-r2}\int_{x0}^{x}[e^{r_1(x-t)}-e^{r_2(x-t)}]b(t)dt \quad (7)$$

Where r_1 and r_2 are the roots of the characteristic polynomial of equation (5), c_1 and c_2 are constants, and $x_0\in I$.

Transforming equation (5) to the form of equation (6) would yield,

$$\frac{d^2\eta}{dx^2}-\frac{\sigma}{\beta}\eta=-\frac{f(x)}{\beta}-\frac{P}{\beta} \tag{8}$$

After setting $l=\frac{\sigma}{\beta}$ and $b(x)=-\frac{P}{\beta}$ since there are no external forces. Hence we obtain,

$$\frac{d^2\eta}{dx^2}-l\eta=b(x) \tag{9}$$

Equation (9) characteristic polynom is

$$\lambda^2-l=0 \tag{10}$$

And its roots are

$$r_{1/2}=\pm\sqrt{l} \tag{11}$$

Now substituting (11) in the equation (7),

$$c_1e^{\sqrt{l}x}+c_2e^{-\sqrt{l}x}+\frac{1}{2\sqrt{l}}\int_{x0}^{x}[e^{\sqrt{l}(x-t)}-e^{-\sqrt{l}(x-t)}]b(t)dt \tag{12}$$

Calculating the integral in (12) yields,

$$c_1e^{\sqrt{l}x}+c_2e^{-\sqrt{l}x}+\frac{b}{2\sqrt{l}}\left[-\frac{e^{\sqrt{l}(x-t)}}{\sqrt{l}}-\frac{e^{-\sqrt{l}(x-t)}}{\sqrt{l}}\right]_{x_0}^{x} \tag{13}$$

Since b is constant.

Setting x_0=0 and substituting the integration boundaries we obtain,

$$c_1e^{\sqrt{l}x}+c_2e^{-\sqrt{l}x}+\frac{b}{2\sqrt{l}}\left(-\frac{1}{\sqrt{l}}+\frac{e^{\sqrt{l}x}}{\sqrt{l}}-\frac{1}{\sqrt{l}}+\frac{e^{-\sqrt{l}x}}{\sqrt{l}}\right) \tag{14}$$

Rearranging elements yields,

$$\psi(x)=c_1e^{\sqrt{l}x}+c_2e^{-\sqrt{l}x}-\frac{b}{l}+\frac{b}{2l}\left(e^{\sqrt{l}x}+e^{-\sqrt{l}x}\right) \tag{15}$$

Setting the boundary conditions $\psi(0)=\psi(1)=0$ yields,

$$\psi(0)=c_1+c_2-\frac{b}{l}+\frac{b}{2l}(1+1)=0\Rightarrow c_1+c_2=0 \tag{16.1}$$

$$\psi(1) = c_1 e^{\sqrt{l}} + c_2 e^{-\sqrt{l}} - \frac{b}{l} + \frac{b}{2l}\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right) = 0 \tag{16.2}$$

Solving equations (16) yields,

$$c_1 = \frac{\frac{b}{l}\left[1 - \frac{\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}} - e^{-\sqrt{l}}} \tag{17.1}$$

$$c_2 = -c_1 \tag{17.2}$$

$$\int_0^1 \psi(x)dx = A_0 \tag{18}$$

$$\int_0^1 \psi(x)dx = \left.\frac{\frac{\frac{b}{l}\left[1 - \frac{\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}} - e^{-\sqrt{l}}} e^{\sqrt{l}x} + \frac{\frac{b}{l}\left[1 - \frac{\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}} - e^{-\sqrt{l}}} e^{-\sqrt{l}\cdot x}}{\sqrt{l}} - \frac{b}{l}x + \frac{b}{2l}\left(\frac{e^{\sqrt{l}x} - e^{-\sqrt{l}x}}{\sqrt{l}}\right)\right|_0^1 \tag{19}$$

$$= \frac{b}{l}\left.\left[\frac{1}{\sqrt{l}}\left(\frac{\left[1 - \frac{\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}} - e^{-\sqrt{l}}}\left(e^{\sqrt{l}x} + e^{-\sqrt{l}\cdot x}\right)\right) - x + \frac{1}{2}\left(\frac{e^{\sqrt{l}x} - e^{-\sqrt{l}x}}{\sqrt{l}}\right)\right]\right|_0^1 = \tag{20}$$

$$= \frac{b}{l}\left[\frac{1}{\sqrt{l}}\left(\frac{\left[1 - \frac{\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}} - e^{-\sqrt{l}}}\left(e^{\sqrt{l}} + e^{-\sqrt{l}\cdot}\right)\right) - 1 + \frac{1}{2}\left(\frac{e^{\sqrt{l}} - e^{-\sqrt{l}}}{\sqrt{l}}\right) - \frac{2\left[1 - \frac{\left(e^{\sqrt{l}} + e^{-\sqrt{l}}\right)}{2}\right]}{\left(e^{\sqrt{l}} - e^{-\sqrt{l}}\right)\sqrt{l}}\right] \tag{21}$$

and from (18) we obtain the relation between the pressure and the initial area,

$$\frac{b}{l}\left[\frac{\dfrac{\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}}-e^{-\sqrt{l}}}e^{\sqrt{l}}+\dfrac{\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}}-e^{-\sqrt{l}}}e^{-\sqrt{l\cdot}}}{\sqrt{l}}-1+\frac{1}{2}\left(\frac{e^{\sqrt{l}}-e^{-\sqrt{l}}}{\sqrt{l}}\right)-\frac{2\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{(e^{\sqrt{l}}-e^{-\sqrt{l}})\sqrt{l}}\right]=A_0 \tag{22}$$

$$b=\frac{A_0 l}{\left[\dfrac{1}{\sqrt{l}}\left(\dfrac{\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}}-e^{-\sqrt{l}}}(e^{\sqrt{l}}+e^{-\sqrt{l\cdot}})\right)-1+\dfrac{1}{2}\left(\dfrac{e^{\sqrt{l}}-e^{-\sqrt{l}}}{\sqrt{l}}\right)-\dfrac{2\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{(e^{\sqrt{l}}-e^{-\sqrt{l}})\sqrt{l}}\right]} \tag{23}$$

$$P=-\left[\frac{A_0 l\beta}{\left[\dfrac{1}{\sqrt{l}}\left(\dfrac{\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{e^{\sqrt{l}}-e^{-\sqrt{l}}}(e^{\sqrt{l}}+e^{-\sqrt{l\cdot}})\right)-1+\dfrac{1}{2}\left(\dfrac{e^{\sqrt{l}}-e^{-\sqrt{l}}}{\sqrt{l}}\right)-\dfrac{2\left[1-\dfrac{\left(e^{\sqrt{l}}+e^{-\sqrt{l}}\right)}{2}\right]}{(e^{\sqrt{l}}-e^{-\sqrt{l}})\sqrt{l}}\right]}\right] \tag{24}$$

Therefore, when A_0 is given, P can be calculated using (24) and the membrane displacement is calculated using (15).

To solve equations (4) and (5) numerically, we need to postulate a relation between, A_0, and either the displacement or the pressure. However we take a more straightforward approach. We first calculate the area under the displaced membrane. Next, since the area under the membrane is proportional to the pressure, a binary search with the pressure as parameter is performed until the requested area is reached.

Thus, the problem represented by equations (4) and (5) reduces to equation (4) above: given a pressure P find the corresponding area under the displaced membrane.

Approximating the solution with the Finite element method will require transforming (4) to the weak form. First multiplying with test function v such that $v \in V_h$ where V_h is

$V_h = \{v : v \in H^1, v(0) = v(1) = 0\}$ and $H^1 = \{v : v \in L_2, \frac{\partial v}{\partial x} \in L_2\}$ and

$L_2 = \{v : \int_0^1 v^2 dx < \infty\}$

$$-\beta(\frac{d^2\eta}{dx^2}, v) + \sigma(\eta, v) - (P, v) = (f, v) \qquad (25)$$

Where $(f, v) = \int_0^1 f \cdot v dx$. Then, discretizing space and approximating η with N basis function φ_i, $\eta = \sum_{i=0}^{N} \xi_i \varphi_i$

$$-\beta(\frac{\sum d^2 \xi_i \cdot \varphi_i}{dx^2}, v) + \sigma(\sum \xi_i \cdot \varphi_i, v) = (f, v) + (P, v) \qquad (26)$$

$$-\beta \sum \xi_i \cdot (\frac{d^2 \varphi_i}{dx^2}, v) + \sigma \sum \xi_i \cdot (\varphi_i, v) = (f, v) + (P, v) \qquad (27)$$

And substituting the basis functions φ_j j=1..N in v transforms (27) to

$$-\beta \sum \xi_i \cdot (\frac{d^2 \varphi_i}{dx^2}, \varphi_j) + \sigma \sum \xi_i \cdot (\varphi_i, \varphi_j) = (f, \varphi_j) + (P, \varphi_j) \qquad j=1..N \qquad (28)$$

By integrating by parts equation (28) and since $v(0) = v(1) = 0$

$$\beta \sum \xi_i \cdot (\frac{d\varphi_i}{dx}, \frac{d\varphi_j}{dx}) + \sigma \sum \xi_i \cdot (\varphi_i, \varphi_j) = (f, \varphi_j) + (P, \varphi_j) \qquad (29)$$

And in matrix form:

$$(\beta \underline{\underline{A}} + \sigma \underline{\underline{B}})\vec{\xi} = \vec{F} + \vec{P} \qquad (30)$$

$$A_{ij} = (\frac{d\varphi_i}{dx}, \frac{d\varphi_j}{dx})$$

Where $B_{ij} = (\varphi_i, \varphi_j)$

$$F_j = (f(x_j), \varphi_j)$$

$$P_j = (P, \varphi_j)$$

Choosing the test functions as linear functions:

$$\varphi_i = \begin{cases} 0 & x < x_{i-1} \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & x_{i-1} < x < x_i \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & x_i < x < x_{i+1} \\ 0 & x > x_{i+1} \end{cases} \qquad (31)$$

Setting $|x_i - x_j| = h$ for $i = j \pm 1$, A and B are:

$A_{ij}=$ $(\frac{d\varphi_i}{dx}, \frac{d\varphi_j}{dx}) = 0$ if j<i-1 or j> i+1

$$(\frac{d\varphi_i}{dx}, \frac{d\varphi_{i+1}}{dx}) = (\frac{d\varphi_i}{dx}, \frac{d\varphi_{i-1}}{dx}) = \int_{x_{i-1}}^{x_i} \frac{1}{h} \cdot -\frac{1}{h} dx = -\frac{1}{h}$$

$$(\frac{d\varphi_i}{dx}, \frac{d\varphi_i}{dx}) = \int_{x_{i-1}}^{x_i} \frac{1}{h^2} dx + \int_{x_i}^{x_{i+1}} \frac{1}{h^2} dx = \frac{2}{h}$$

$B_{ij} =$

$(\varphi_i, \varphi_j) = 0$ if j<i-1 or j> i+1

$$(\varphi_i, \varphi_{i+1}) = (\varphi_i, \varphi_{i-1}) = \frac{1}{h^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})(x_i - x) dx =$$

$$\frac{1}{h^2} \left[\frac{(x - x_{i-1})^2}{2} (x_i - x) \Big|_{x_{i-1}}^{x_i} - \int_{x_{i-1}}^{x_i} (-1) \frac{(x - x_{i-1})^2}{2} \right] = \frac{(x - x_{i-1})^3}{6h^2} \Bigg|_{x_{i-1}}^{x_i} = h/6$$

$$(\varphi_i, \varphi_i) = \frac{1}{h^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 dx + \frac{1}{h^2} \int_{x_{i_i}}^{x_{i+1}} (x_{i+1} - x)^2 dx = \frac{(x - x_{i-1})^3}{3h^2} \Bigg|_{x_{i-1}}^{x_i} - \frac{(x_{i+1} - x)^3}{3h^2} \Bigg|_{x_i}^{x_{i+1}} = 2h/3$$

So $A_{ij} = (\frac{\partial \varphi_i}{\partial x}, \frac{\partial \varphi_j}{\partial x}) = \begin{cases} 0 & j < i-1 \\ -\frac{1}{h} & i = j \pm 1 \\ \frac{2}{h} & i = j \\ 0 & j > i+1 \end{cases}$ and $B_{ij} = (\varphi_i, \varphi_j) = \begin{cases} 0 & j < i-1 \\ \frac{h}{6} & i = j \pm 1 \\ \frac{2h}{3} & i = j \\ 0 & j > i+1 \end{cases}$

And the steady-state solution is:

$$\vec{\xi} = (\beta A + \sigma B)^{-1}(\vec{F} + \vec{P}) \qquad (32)$$

With equations (4), (5) we now have the final algorithm for calculating the membrane displacement.

Algorithm 1.

1. Set P_{low}=0 P_1 = initial guess pressure (any number greater than 0), P_{high} = Much Larger number than P_1.
2. set i =1.
3. While $|V^c – V_0|$ > Tolerance
   a. Calculate the displacement of the membrane displacement using the numerical scheme with $\vec{F} = 0$ and $\vec{P} = P_i \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$
   b. Set $V^c = \int \xi dx$
   c. If $V^c – V_0$ > Tolerance
      i. Set $P_{i+1} = (P_i+P_{low})/2$
      ii. $P_{high}=P_i$
   d. If $V^c – V_0$ < -Tolerance
      i. Set $P_{i+1} = (P_i+P_{high})/2$
      ii. $P_{low} = P_i$
   e. i = i+1

Note

- The input to Algorithm 1 is the initial area, A_0, and the tolerance level, tolerance, which determines the error of the algorithm

Problem 2.a - ideal fluid

An ideal fluid has no friction; therefore, it serves only as area conservater.
The equations that control the membrane shape are:

$$\frac{\partial^2 \eta}{\partial t^2} - \beta \frac{\partial^2 \eta}{\partial x^2} + \sigma\eta - P = f \tag{33}$$

$$\int \eta dx = V_0 \tag{34}$$

Where $\eta = \eta(x,t)$ is the membrane displacement, V_0 is the initial area, β and σ are properties of the membrane, f(x,t) is the external and P is the Lagrange multiplier for constraint (34).

Transforming to the weak form by multiplying with a test $v \in V_h$ function, integrating and setting $\eta = \sum_{i=0}^{N} \xi_i(t)\varphi_i$ where $\varphi_i \in V_h$, yields:

$$(\frac{\partial^2 \eta}{\partial t^2}, v) - \beta(\frac{\partial^2 \eta}{\partial x^2}, v) + \sigma(\eta, v) = (f, v) + (P, v) \tag{35}$$

$$(\sum \frac{\partial^2 \xi_i(t) \cdot \varphi_i}{\partial t^2}, v) - \beta(\sum \frac{\xi_i(t) \cdot \partial^2 \varphi_i}{\partial x^2}, v) + \sigma(\sum \xi_i(t) \cdot \varphi_i, v) = (f, v) + (P, v) \tag{36}$$

$$\sum \frac{\partial^2 \xi_i(t)}{\partial t^2} \cdot (\varphi_i, v) - \beta \sum \xi_i(t) \cdot (\frac{\partial^2 \varphi_i}{\partial x^2}, v) + \sigma \sum \xi_i(t) \cdot (\varphi_i, v) = (f, v) + (P, v) \qquad (37)$$

Replacing v with the basis function φ_j j=1..N yields,

$$\sum \frac{\partial^2 \xi_i(t)}{\partial t^2} \cdot (\varphi_i, \varphi_j) - \beta \sum \xi_i(t) \cdot (\frac{\partial^2 \varphi_i}{\partial x^2}, \varphi_j) + \qquad \text{j=1..N} \qquad (38)$$

$$\sigma \sum \xi_i(t) \cdot (\varphi_i, \varphi_j) = (f, v) + (P, v)$$

And after integration by parts we obtain,

$$\sum \frac{\partial^2 \xi_i(t)}{\partial t^2} \cdot (\varphi_i, \varphi_j) + \beta \sum \xi_i \cdot (\frac{\partial \varphi_i}{\partial x}, \frac{\partial \varphi_j}{\partial x}) + \sigma \sum \xi_i \cdot (\varphi_i, \varphi_j) = (f, v) + (P, v) \qquad (39)$$

Discretizing time and using the implicit forward-backward scheme to solve the second derivative yields,

$$\underline{\underline{B}} \left(\frac{\vec{\xi}^{k+1} - 2\vec{\xi}^{k} + \vec{\xi}^{k-1}}{\delta t^2} \right) + \beta \underline{\underline{A}} \vec{\xi}^{k+1} + \sigma \underline{\underline{B}} \vec{\xi}^{k+1} = \vec{F}^{k+1} + \vec{P}^{k+1} \qquad (40)$$

where $\vec{\xi}^{k} = \begin{pmatrix} \xi_1^k \\ \vdots \\ \xi_N^k \end{pmatrix}$, $\vec{F}^{k+1} = \begin{pmatrix} (f(t_k), \varphi_1) \\ \vdots \\ (f(t_k), \varphi_N) \end{pmatrix}$, $\vec{P}^{k+1} = \begin{pmatrix} (P^{k+1}, \varphi_1) \\ \vdots \\ (P^{k+1}, \varphi_N) \end{pmatrix}$

where t_k is the time at time step k.

Lumping the mass matrix would give a good approximation and would simplify calculations.

So the equations are:

$$\underline{\underline{B}} \cdot \vec{\xi}^{k+1} + \delta t^2 \cdot \beta \cdot \underline{\underline{A}} \cdot \vec{\xi}^{k+1} + \delta t^2 \cdot \sigma I \cdot \vec{\xi}^{k+1} = \delta t^2 \cdot (\vec{F}^{k+1} + \vec{P}^{k+1}) + 2I \cdot \vec{\xi}^{k} - I \cdot \vec{\xi}^{k-1} \qquad (41)$$

Reorganizing equation (22) yields,

$$(\delta t^2 \cdot (\beta \underline{\underline{A}} + \sigma I) + 1) \cdot \vec{\xi}^{k+1} = \delta t^2 \cdot \vec{F}^{k+1} + 2I \cdot \vec{\xi}^{k} - I \cdot \vec{\xi}^{k-1} + \delta t^2 \cdot \vec{P}^{k+1} \tag{42}$$

setting: $\underline{\underline{C}} = (\delta t^2 \cdot (\beta \underline{\underline{A}} + \sigma I) + 1)$ yields,

$$\underline{\underline{C}} \cdot \vec{\xi}^{k+1} = \delta t^2 \cdot \vec{F}^{k+1} + 2I \cdot \vec{\xi}^{k} - I \cdot \vec{\xi}^{k-1} + \delta t^2 \cdot \vec{P}^{k+1} \tag{43}$$

Since in (26) both P and ξ are unknowns, an additional equation, the mass conservation equation (equation 15), is also employed. Now the relation between the two must be formulated.

Rearranging the elements in the right wing of equation (26) and introducing two new variables results in,

$$\underline{\underline{C}} \cdot \left(\vec{\xi}^{k+1}_{external} + \vec{\xi}^{k+1}_{INTERNAL} \right) = \delta t^2 \cdot \vec{P}^{k+1} + \left(\delta t^2 \cdot f(t_k) + 2I \cdot \vec{\xi}^{k} - I \cdot \vec{\xi}^{k-1} \right) \tag{44}$$

Therefore

$$\left(\vec{\xi}^{k+1}_{external} + \vec{\xi}^{k+1}_{INTERNAL} \right) = \vec{\xi}^{k+1}. \tag{45}$$

Since the pressure serves as a Lagrange multiplier for equation (15) it is a scalar with constant value at each node or a constant vector $\vec{P}^{k+1} = \alpha^{k+1} \cdot \vec{1}$.

Setting the $\vec{\xi}^{k+1}_{INTERNAL}$ and $\vec{\xi}^{k+1}_{external}$ to be

$$\vec{\xi}^{k+1}_{external} = \underline{\underline{C}}^{-1}(\delta t^2 \cdot f(t_{k+1}) + 2I \cdot \vec{\xi}^{k} - I \cdot \vec{\xi}^{k-1}) \tag{46}$$

$$\vec{\xi}^{k+1}_{INTERNAL} = \alpha^{k+1} \cdot \delta t^2 \cdot \underline{\underline{C}}^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \tag{47}$$

Setting

$$\vec{\xi}^{k+1}_{\mathrm{int}\,ernal} = \delta t^2 \cdot \underline{\underline{C}}^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

and from (46) and (47) :

$$\vec{\xi}^{k+1}_{INTERNAL} = \alpha^{k+1} \cdot \vec{\xi}^{k+1}_{\mathrm{int}\,ernal} \,. \tag{48}$$

from (45)

$$\vec{\xi}^{k+1}_{external} + \alpha^{k+1} \cdot \vec{\xi}^{k+1}_{\mathrm{int}\,ernal} = \vec{\xi}^{+1} \tag{49}$$

and

$$\int_0^1 \xi^{k+1} dx = \int_0^1 \xi^{k+1}_{external} dx + \alpha^{k+1} \cdot \int_0^1 \xi^{k+1}_{\mathrm{int}\,ernal} dx = V_0 \tag{50}$$

The unknown α^{k+1} can be calculated from equation (50) ,

$$\alpha^{k+1} = \frac{V_0 - \int_0^1 \xi^{k+1}_{external} dx}{\int_0^1 \xi^{k+1}_{\mathrm{int}\,ernal} dx} \tag{51}$$

With equations (14), (15) we now have the final algorithm for calculating the membrane displacement.

Algorithm 2:

1. Set $\vec{\xi}_{internal} = \delta t^2 \cdot \underline{\underline{C}}^{-1} \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$
2. For each time step k.
    a. Calculate ξ^{k+1}_{Force} using (43).
    b. Find α^{k+1} using (51).
    c. Calculate ξ^{k+1} using (49).

Note

- Algorithm 2 receives as input two conformations of the membrane ξ^0, ξ^1 where ξ^0 is the membrane conformation at the first time step and ξ^1 is the membrane conformation at the second time step. To solve problem 2.a set $\xi^0 = \xi^1 =$ steady-state solution.

Problem 2.b - viscous fluid

Since viscous fluid has friction with the membrane, the fluid serves not only as a area conservator but also carry force on the membrane and thus the internal forces are a combination of the pressure and stress force. The stress tensor of newtonian fluid is derived from the velocity field. Since the fluid stick to the membrane its movement changes the velocity field.

The equations that govern the flow of an incompressible viscous fluid are Navier-Stokes equations:

$$\rho_0 (\frac{\partial v}{\partial t} + (v \cdot \nabla) v) - \mu \Delta v + \nabla p = 0 \qquad (52)$$

$$\nabla v = 0 \qquad (53)$$

Where p is the pressure and v is the velocity vector, μ is the dynamic viscosity and ρ is the density.

With the two boundary conditions:

$$v = \frac{\partial \eta}{\partial t} e_y \text{ for } x \in \Gamma_{membrane} \text{ (see Figure 1)} \qquad (54)$$

$$v = 0 \text{ for } x \in \Gamma_{solid} \text{ (see Figure 1)} \qquad (55)$$

Where η is the membrane displacement and e_y is the y axis.

Solving the problem is done by splitting the problem into two sub problems:

<u>Sub problem 1:</u>

Find the membrane displacement, given the external and internal forces.

<u>Sub problem 2:</u>

Find the velocity field at a given the membrane displacement.

Sub problem 1

Solving sub problem 1 is similar to solving equation (14) and (15) (problem 2a) with the exception that now the internal force on the membrane is the traction

$$\underline{T} = \underline{\underline{\sigma}} \cdot \underline{n} \tag{56}$$

Where $\underline{\underline{\sigma}}$ is the stess-tensor. $\underline{\underline{\sigma}} = -pI + S$. I is the identity matrix, pI is the pressure tensor and S is extra-stress tensor. At Newtonian fluid S is linear with the strain-tensor, that is, $S = 2\mu d = \mu(\nabla u + \nabla u^T)$. Here we take the external forces to be, as in problem 2a, a given function of time and space.

Sub problem 2

Since the problem is 2D then setting $v = (u, v)$ and discretizing Navier-Stokes would provide,

$$\begin{cases} \rho_0\left(\dfrac{\partial u}{\partial t} + u \cdot \dfrac{\partial u}{\partial x} + v \cdot \dfrac{\partial u}{\partial y}\right) - \mu\dfrac{\partial^2 u}{\partial x^2} - \mu\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial p}{\partial x} = 0 \\ \rho_0\left(\dfrac{\partial v}{\partial t} + u \cdot \dfrac{\partial v}{\partial x} + v \cdot \dfrac{\partial v}{\partial y}\right) - \mu\dfrac{\partial^2 v}{\partial x^2} - \mu\dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial p}{\partial y} = 0 \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases} \tag{57}$$

Setting $u = \sum_{i=0}^{N} U_i \varphi_i$, $v = \sum_{i=0}^{N} V_i \varphi_i$ and $p = \sum_{i=0}^{N} p_i \psi_i$ and multiplying with test functions g, l $\in V_h^{2D}$ where $V_h^{2D} = \{v : v \in H^1, v|_{\partial\Omega} = 0\}$ and $H^1 = \{v : v \in L_2^{2D}, \dfrac{\partial v}{\partial x} \in L_2^{2D}, \dfrac{\partial v}{\partial y} \in L_2^{2D}\}$ where $L_2^{2D} = \{v : \int_\Omega v^2 dx < \infty\}$ and integrating on the domain Ω yields,

$$\begin{cases} \rho_0\left(\sum_{i=0}^{N}\frac{\partial U_i}{\partial t}(\varphi_i, g) + \sum_{i=0}^{N}U_i(\frac{\partial \varphi_i}{\partial x}, u\cdot g) + \sum_{i=0}^{N}U_i(\frac{\partial \varphi_{i,}}{\partial y}, v\cdot g)\right) \\ -\mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial^2 \varphi_i}{\partial x^2}, g) - \mu\sum_{i=0}^{N}U_i\cdot(\frac{\partial^2 \varphi_i}{\partial y^2}, g) + \sum_{k=0}^{M}P_i\cdot(\frac{\partial \psi_i}{\partial x}, l) = 0 \\ \\ \rho_0\left(\sum_{i=0}^{N}\frac{\partial V_i}{\partial t}(\varphi_i, g) + \sum_{i=0}^{N}V_i(\frac{\partial \varphi_i}{\partial x}, u\cdot g) + \sum_{i=0}^{N}V_i(\frac{\partial \varphi_{i,}}{\partial y}, v\cdot g)\right) \\ -\mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial^2 \varphi_i}{\partial x^2}, g) - \mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial^2 \varphi_i}{\partial y^2}, g) + \sum_{k=0}^{N}P_i\cdot(\frac{\partial \psi_i}{\partial y}, l) = 0 \\ \\ \sum_{i=0}^{N}U_i\cdot(\frac{\partial \varphi_i}{\partial x}, l) + \sum_{i=0}^{N}V_i\cdot(\frac{\partial \varphi_i}{\partial y}, l) = 0 \end{cases} \tag{58}$$

Setting g = φ_j j = 1...N and l=ψ_j j=1..N yields

$$\begin{cases} \rho_0\left(\sum_{i=0}^{N}\frac{\partial U_i}{\partial t}(\varphi_i, \varphi_j) + \sum_{i=0}^{N}U_i(\frac{\partial \varphi_i}{\partial x}, u\cdot \varphi_j) + \sum_{i=0}^{N}U_i(\frac{\partial \varphi_{i,}}{\partial y}, v\cdot \varphi_j)\right) \\ -\mu\sum_{i=0}^{N}U_i\cdot(\frac{\partial^2 \varphi_i}{\partial x^2}, \varphi_j) - \mu\sum_{i=0}^{N}U_i\cdot(\frac{\partial^2 \varphi_i}{\partial y^2}, \varphi_j) + \sum_{k=0}^{N}P_i\cdot(\frac{\partial \psi}{\partial x}, \varphi_j) = 0 \\ \\ \rho_0\left(\sum_{i=0}^{N}\frac{\partial V_i}{\partial t}(\varphi_i, \varphi_j) + \sum_{i=0}^{N}V_i(\frac{\partial \varphi_i}{\partial x}, u\cdot \varphi_j) + \sum_{i=0}^{N}V_i(\frac{\partial \varphi_{i,}}{\partial y}, v\cdot \varphi_j)\right) \\ -\mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial^2 \varphi_i}{\partial x^2}, \varphi_j) - \mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial^2 \varphi_i}{\partial y^2}, \varphi_j) + \sum_{k=0}^{N}P_i\cdot(\frac{\partial \psi}{\partial y}, \varphi_j) = 0 \\ \\ \sum_{i=0}^{N}U_i\cdot(\frac{\partial \varphi_i}{\partial x}, \psi_j) + \sum_{i=0}^{N}V_i\cdot(\frac{\partial \varphi_i}{\partial y}, \psi_j) = 0 \end{cases} \tag{59}$$

Integrating by parts (43) leads to (44) because the test functions equal 0 on $\Gamma_{membrane} \cup \Gamma_{solid}$,

$$\begin{cases} \rho_0\left(\sum_{i=0}^{N}\frac{\partial U_i}{\partial t}(\varphi_i,\varphi_j)+\sum_{i=0}^{N}U_i(\frac{\partial\varphi_i}{\partial x},u\cdot\varphi_j)+\sum_{i=0}^{N}U_i(\frac{\partial\varphi_{i,}}{\partial y},v\cdot\varphi_j)\right) \\ -\mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial\varphi_i}{\partial x},\frac{\partial\varphi_j}{\partial x})-\mu\sum_{i=0}^{N}U_i\cdot(\frac{\partial\varphi_i}{\partial y},\frac{\partial\varphi_j}{\partial y})-\sum_{k=0}^{N}P_i\cdot(\frac{\partial\varphi_i}{\partial x},\psi_j)=0 \\ \\ \rho_0\left(\sum_{i=0}^{N}\frac{\partial V_i}{\partial t}(\varphi_i,\varphi_j)+\sum_{i=0}^{N}V_i(\frac{\partial\varphi_i}{\partial x},u\cdot\varphi_j)+\sum_{i=0}^{N}V_i(\frac{\partial\varphi_{i,}}{\partial y},v\cdot\varphi_j)\right) \\ -\mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial\varphi_i}{\partial x},\frac{\partial\varphi_j}{\partial x})-\mu\sum_{i=0}^{N}V_i\cdot(\frac{\partial\varphi_i}{\partial y},\frac{\partial\varphi_j}{\partial y})-\sum_{k=0}^{N}P_i\cdot(\frac{\partial\varphi_i}{\partial y},\psi_j)=0 \\ \\ \sum_{i=0}^{N}U_i\cdot(\frac{\partial\varphi_i}{\partial x},\psi_j)+\sum_{i=0}^{N}V_i\cdot(\frac{\partial\varphi_i}{\partial y},\psi_j)=0 \end{cases} \tag{60}$$

The matrix notation of (60) would be:

$$\begin{cases} M\frac{\partial\vec{U}}{\partial t}+(A_1(u)+A_2(u))\vec{U}-\mu(K_{11}\vec{U}+K_{22}\vec{U})-C_1\vec{P}=0 \\ M\frac{\partial\vec{V}}{\partial t}+(A_1(u)+A_2(u))\vec{V}-\mu(K_{11}\vec{V}+K_{22}\vec{V})-C_2\vec{P}=0 \\ C_1\vec{U}+C_2\vec{V}=0 \end{cases} \tag{61}$$

Where

$$M_j=\rho_0\int_\Omega\sum_i\varphi_i\varphi_j dV \qquad C_{1j}=\int_\Omega\sum_i\frac{\partial\varphi_i}{\partial x}\psi_j dV$$

$$A_{1j}(u)=\rho_0\int_\Omega\sum_i u\cdot\frac{\partial\varphi_i}{\partial x}\varphi_j dV \qquad C_{2j}=\int_\Omega\sum_i\frac{\partial\varphi_i}{\partial x}\psi_j dV$$

$$A_{2j}(u)=\rho_0\int_\Omega\sum_i v\cdot\frac{\partial\varphi_i}{\partial x}\varphi_j dV \qquad \vec{U}=\sum_{i=1}^{N}U_i$$

$$K_{11j}=\int_\Omega\sum_i\frac{\partial\varphi_i}{\partial x}\frac{\partial\varphi_j}{\partial x}dV \qquad \vec{V}=\sum_{i=1}^{N}V_i$$

$$K_{22j}=\int_\Omega\sum_i\frac{\partial\varphi_i}{\partial y}\frac{\partial\varphi_j}{\partial y}dV$$

Recollecting variables, we obtain,

$$\begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial \vec{U}}{\partial t} \\ \frac{\partial \vec{V}}{\partial t} \\ \frac{\partial \vec{P}}{\partial t} \end{pmatrix} + \\ + \begin{pmatrix} A_1(u) + A_2(u) + \mu(K_{11} + K_{22}) & 0 & C1T \\ 0 & A_1(u) + A_2(u) + \mu(K_{11} + K_{22}) & C2T \\ C1 & C2 & 0 \end{pmatrix} \begin{pmatrix} \vec{U} \\ \vec{V} \\ \vec{P} \end{pmatrix} = \begin{pmatrix} \vec{0} \\ \vec{0} \\ \vec{0} \end{pmatrix} \quad (62)$$

At each time step the displacement of the membrane is calculated using the forces imposed on it during the preceding time step, and then the velocities are calculated using (63) (Fig 2).

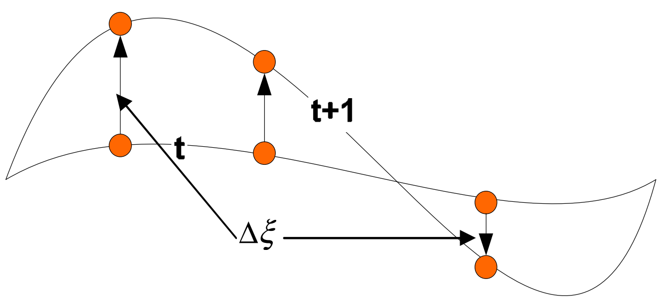

Figure 2 – Calculating the velocities of the nodes on the membrane

The velocity on node n is,

$$v\Big|_n = \frac{\xi^{t+1} - \xi^t}{\delta t}\Bigg|_n \quad (63)$$

Since the elasticity problem is more sensitive than the flow problem, the elasticity problem is solved with smaller time steps than the flow problem, updating the stress tensor only once in K time steps.

With equations (52), (53), (54), (55), (14), (15) we now have the final algorithm for calculating the membrane displacement.

Algorithm 3:

Set $\xi^0 = \xi^1 =$ steady-state solution.

1. Set stress forces, $\vec{F} = 0$.
2. set $\vec{\xi}^{temporary} = \vec{\xi}^{0}$
3. For each time step k.
   a. For i = 1 to K
      i. Calculate $\vec{\xi}^{i+1}$ using Algorithm 2 with $\vec{\xi}^{temporary}, \vec{\xi}^{k-1}$ as the membrane position parameters, the stress forces $\vec{F}$ and the external forces (see notes).
   b. Set $\vec{\xi}^{k} = \vec{\xi}^{K+1}$, $\vec{\xi}^{temporary} = \vec{\xi}^{K}$
   c. Calculate node velocities with (63) using $\vec{\xi}^{k+1}$ and $\vec{\xi}^{k}$
   d. Solve fluid problem with boundary conditions using (62) subject to boundary conditions (54) and (55) with the velocities calculated at b.
   e. Calculate the stress forces and update $\vec{F}$.

Notes

- The time step variable, k, starts at 2.
- We assume that the external force satisfies the condition f(0) =0, and that it is continuous.
- Since the traction forces are functions of velocity, and at steady-state the velocity is 0 then the stress forces at t=0 are zero.
- At the first time step the force imposed by the fluid is only the hydrostatic. This is calculated from the steady-state conditions.

Results

All the problems were solved numerically, using FIDAP version 8.6.2 for the numerical solution of the fluid problem and MATLAB R12 for the elasticity problem.
The units system is KGS.

The domain dimensions are L=1 meter wide and H=0.2 meter height. The dynamic
The element width, h, is 0.01m. The elasticity time step is 0.01 sec and the flow time step is 0.1sec.

Problem 1 – steady-state

Figure 3 shows the solutions of the steady-state problem of membranes with different properties as was calculated by algorithm 1 with the analytical solution of equation (5). The pressure was adjusted to maintain an area of $1m^2$.

Membrane properties	Pressure $N/m^2 = Pa$ required maintaining a constant area of 1 m^2).
β=1, σ=1 (Blue)	13.1986
β=100, σ=1 (Red)	1201.2
β=1, σ=100 (Green)	124.9972

Table 2 –The pressure required to maintain an area of $1m^2$ under the membrane with various properties.

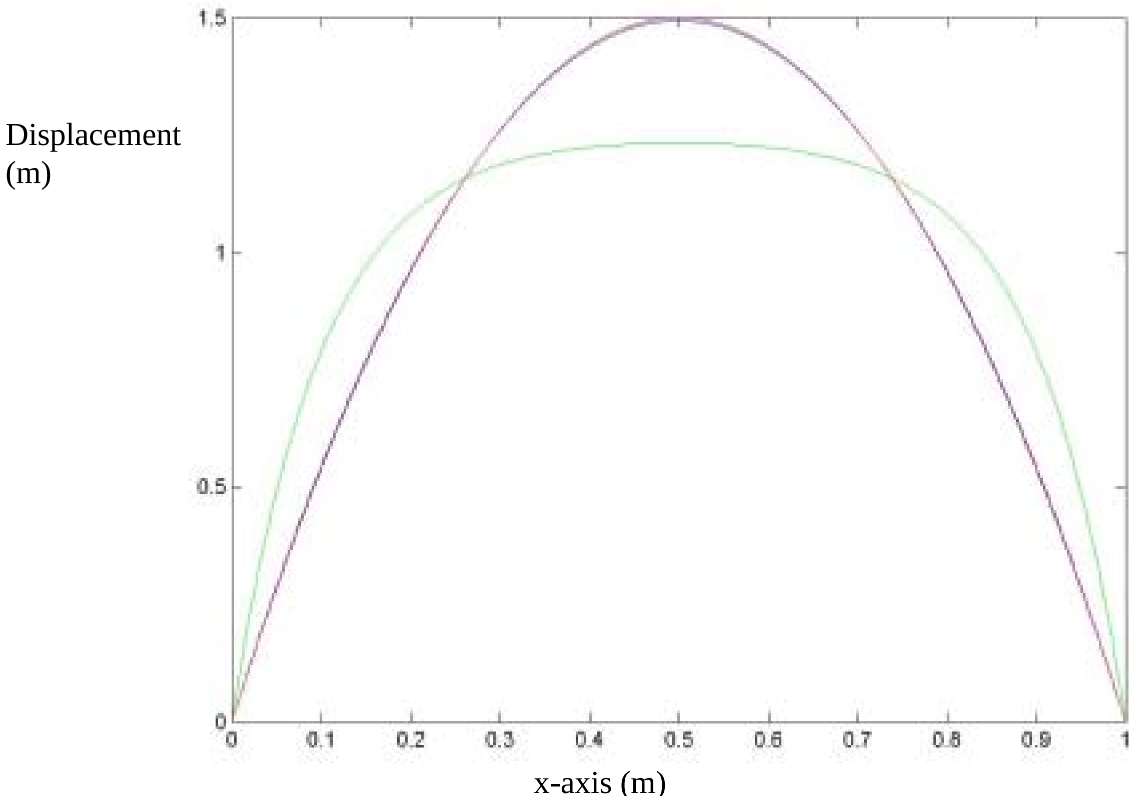

Figure 3 Simulation results for problem 1 when a constant area of 1m^2 was maintained under the membrane with various properties as in Table 1.

Table 2 and Figure 17 show that β greatest influence is in determining the force that has to be applied on the membrane while σ determines the membranes shape.

Problem 2 – ideal and viscous fluids

Figures 4-7 show the kinetics of membrane displacement at three different tests after imposing the external pressure f(t) (in. $N/m^2 = Pa$) given by equation ,

$$f(x,t) = \begin{cases} 0 \le x \le 0.3, 0.05 \le t \le 2 & -150 \cdot x^2 \cdot (t-0.05) \\ else & 0 \end{cases} \tag{48}$$

At each test the membrane was filled with fluid with different viscosity:

1. Test 1 – The fluid has no viscosity (ideal fluid).
2. Test 2 - The fluid has viscosity of 1 $N \cdot Sec/m^2$.
3. Test 3 - The fluid has viscosity of 10 $N \cdot Sec/m^2$.

The forces is inflicted on the membrane at the steady-state displacement after being distended by 0.2m^2 of fluid.

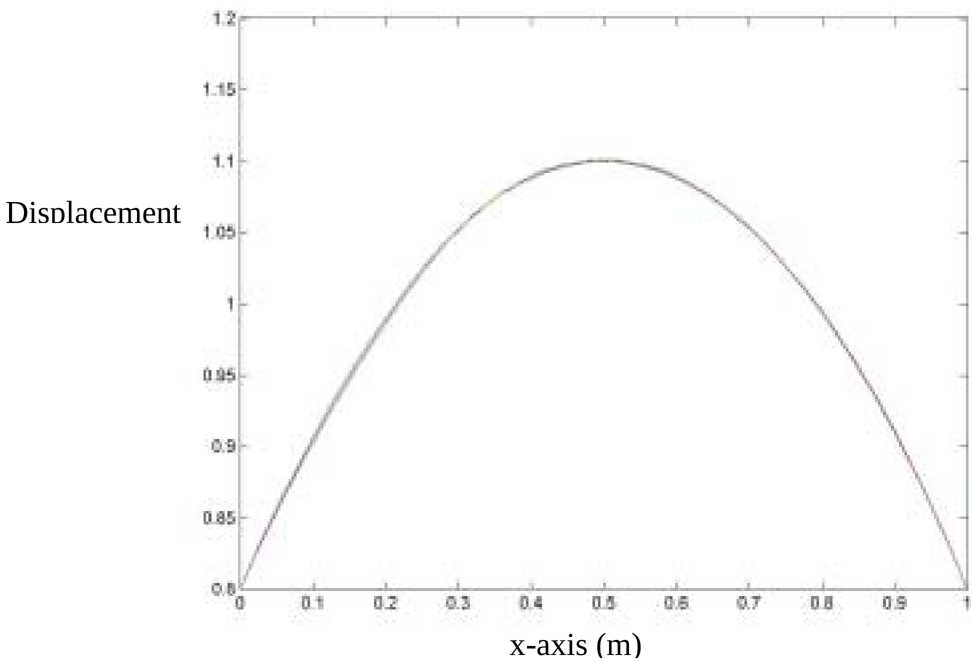

Figure 4 Membrane shape at t=0sec for tests 1 (blue), 2 (red) and 3 (green).

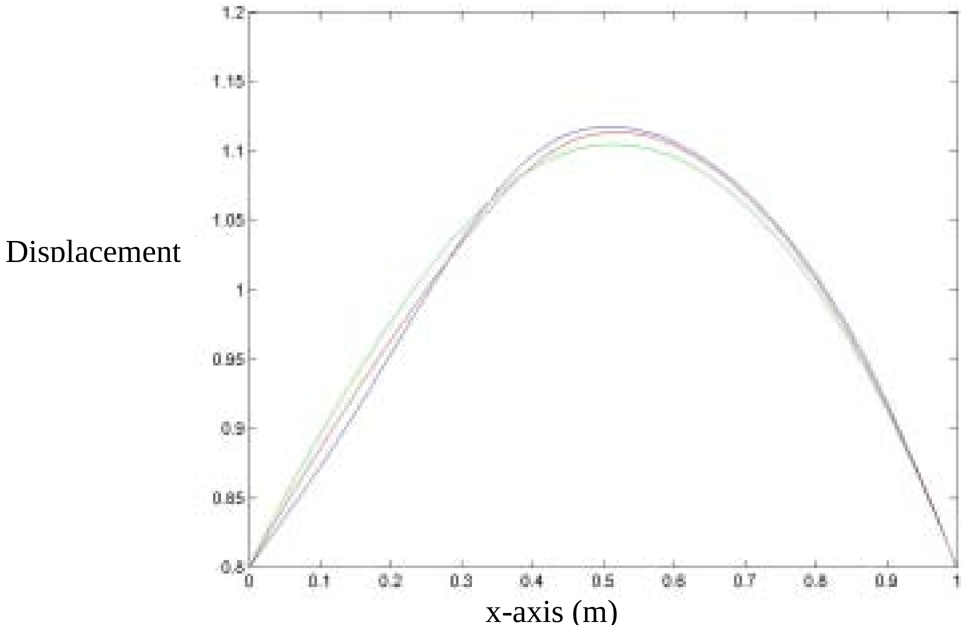

Figure 5 Membranes shape at t=2sec for tests 1, 2 and 3 (as in figure 20)

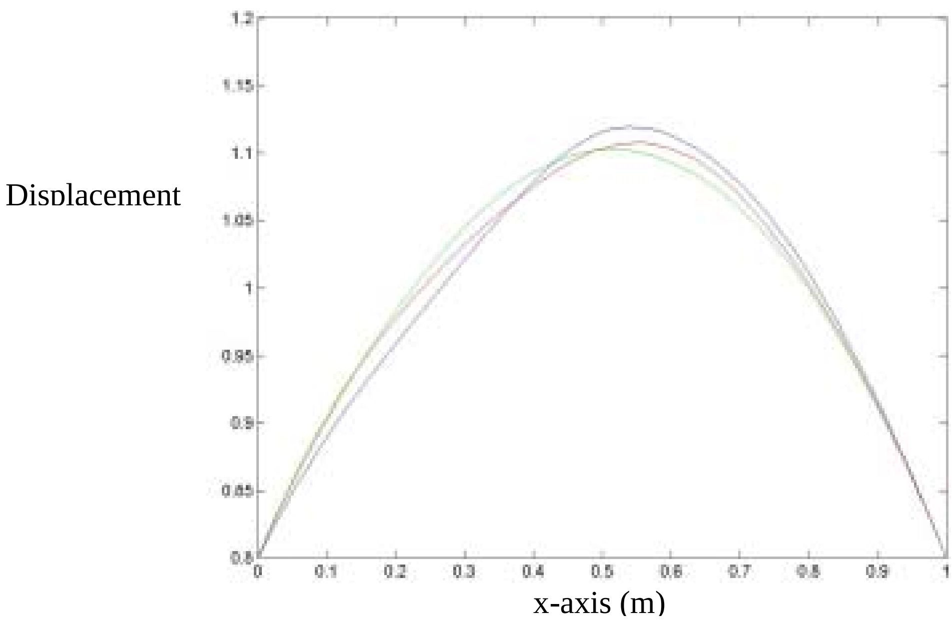

Figure 6 Membranes shape at t=3sec for tests 1, 2 and 3 (as in figure 20)

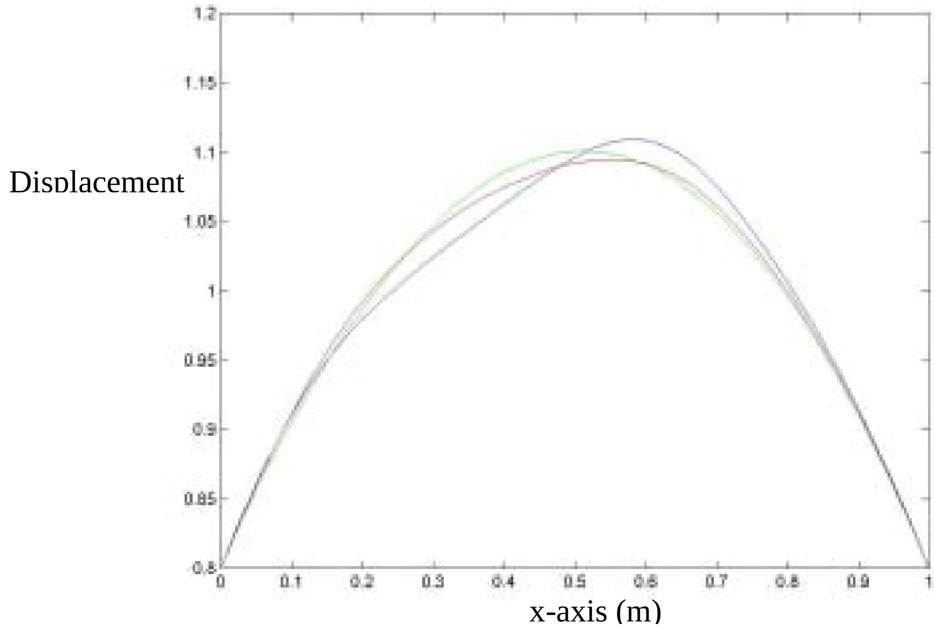

Figure 7 Membrane shape at time t=4 for tests 1, 2 and 3 (as in figure 20)

High viscosity dissipates the external force exerted on the fluid and therefore at high viscosity the membrane displacement is smaller (Figures 4-7).

Figures8-17 show the velocity field and the stream lines induced by the membrane deformation at different time points in test 1.

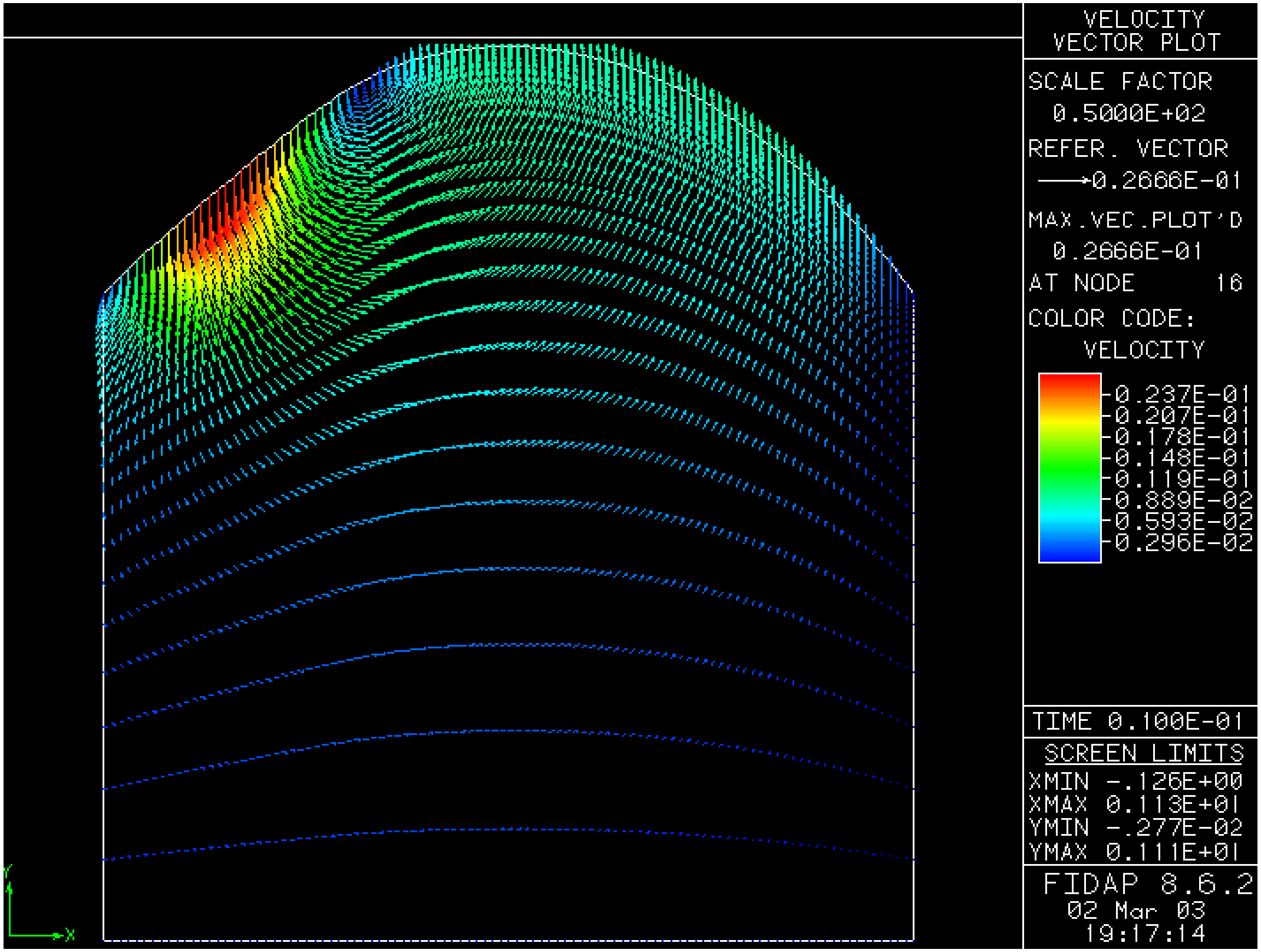

Figure 8 Velocity field at t=2sec

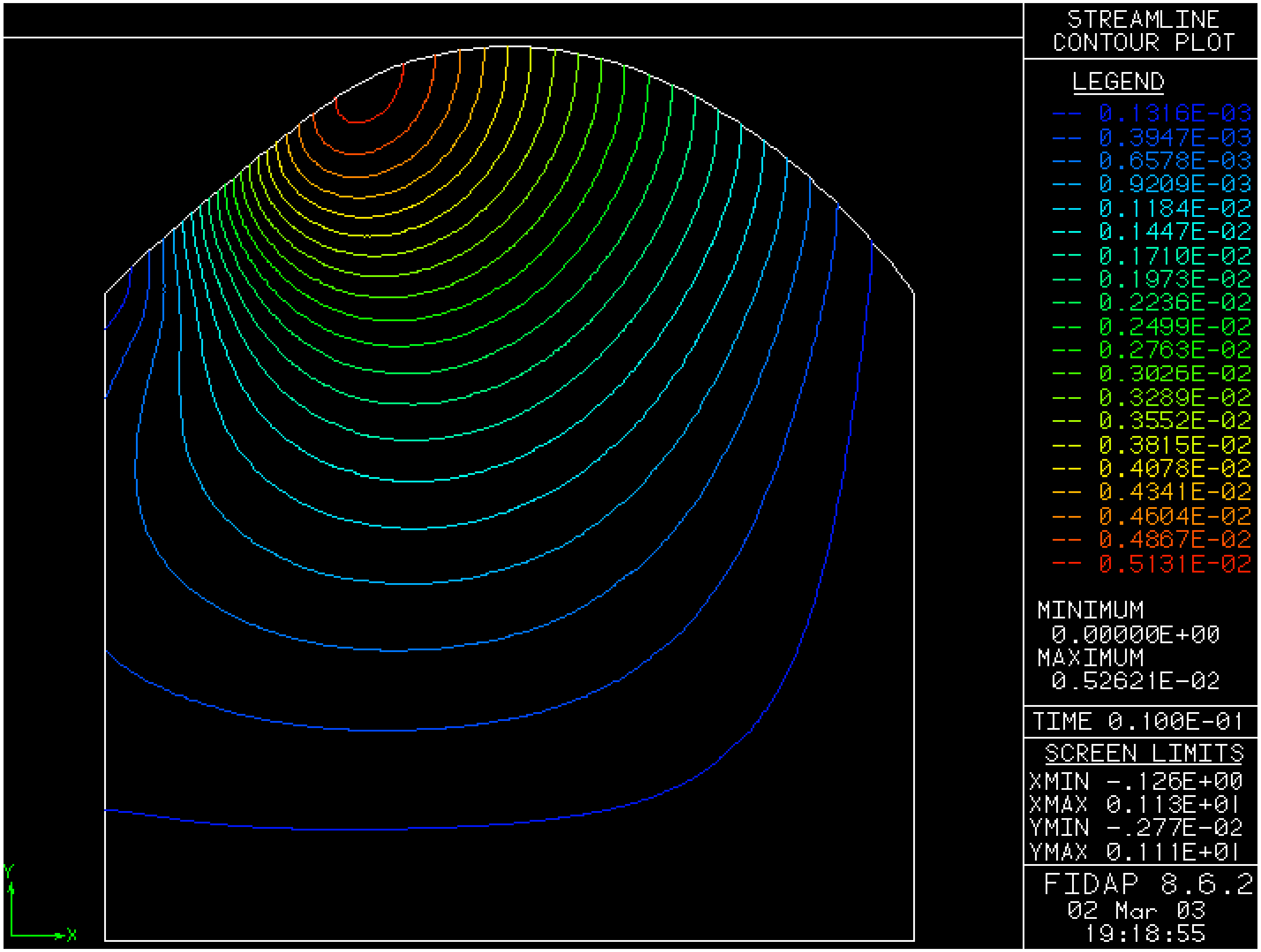

Figure 9 stream lines at=2sec

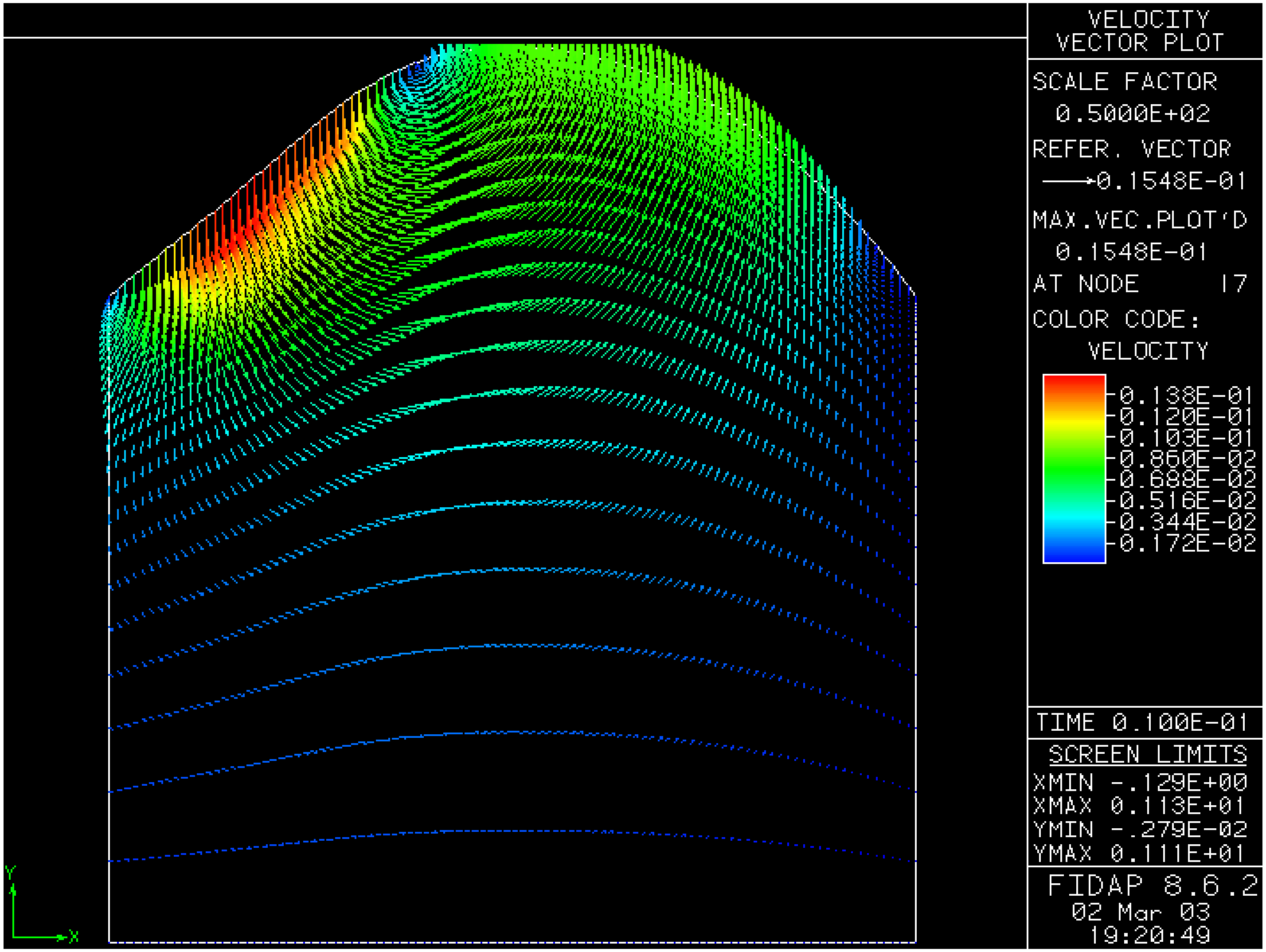

Figure 10 vector field at t=2.5 sec

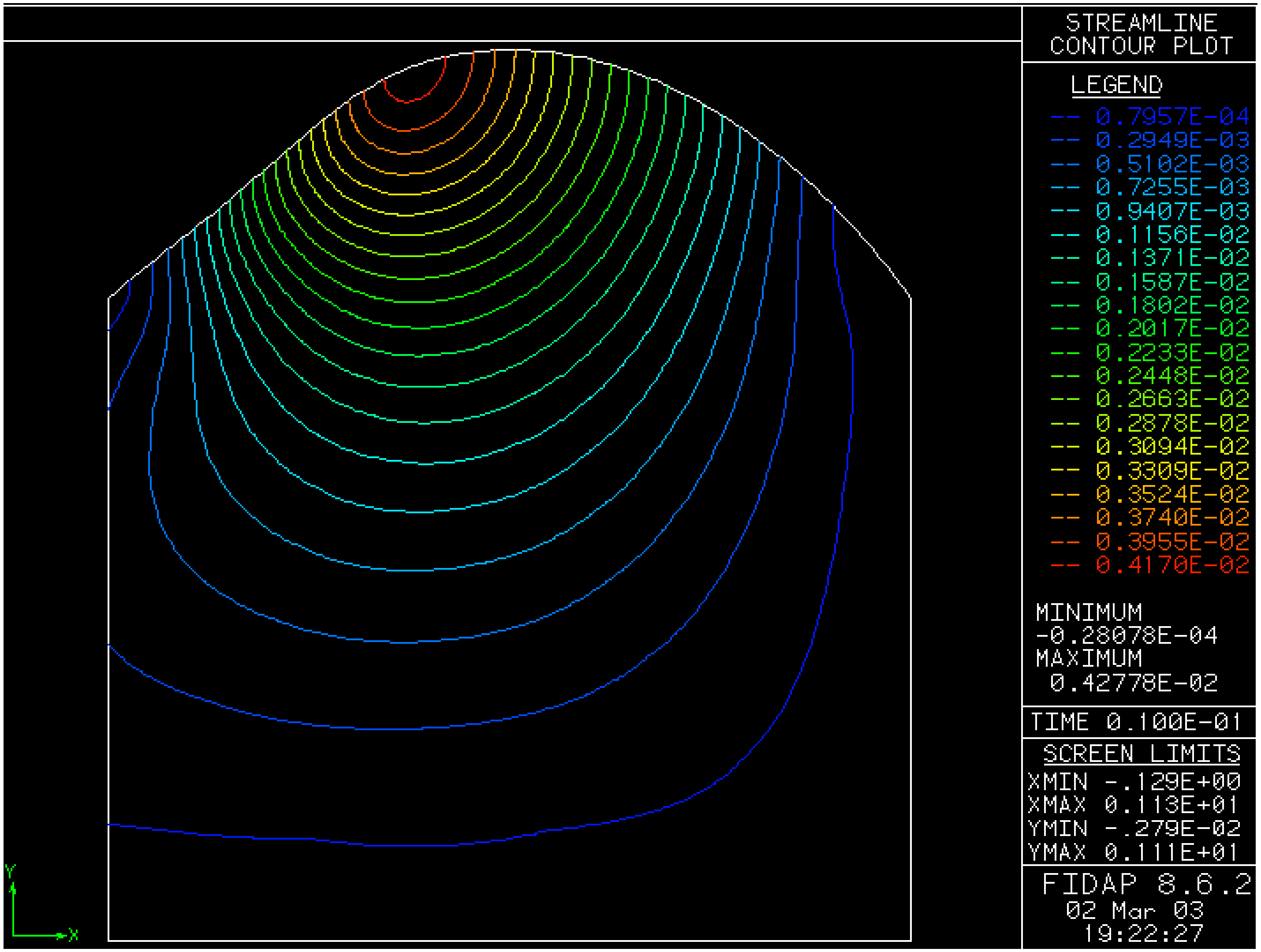

Figure 11 stream lines at t=2.5sec

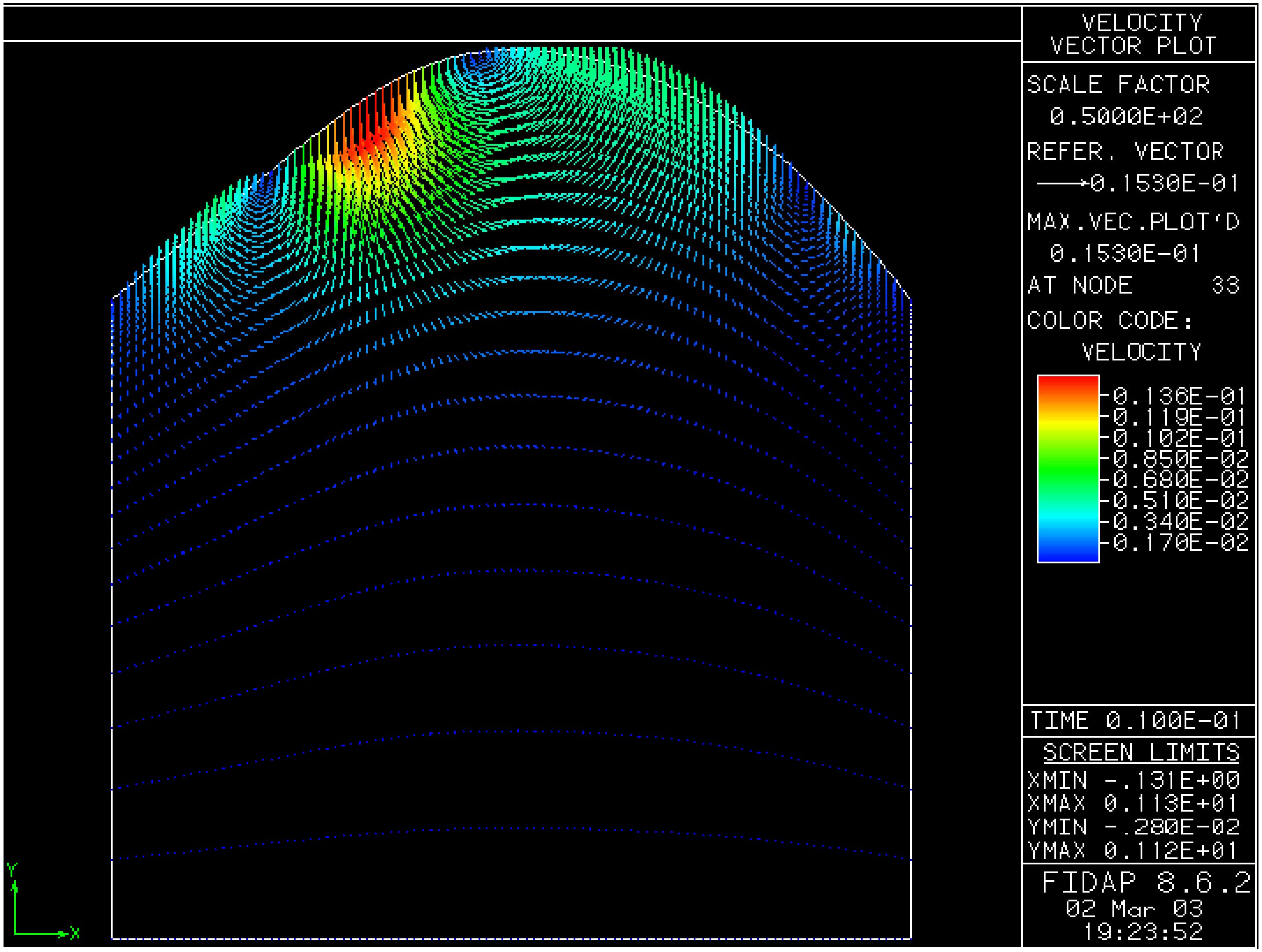

Figure 12 Velocity field at t=3sec

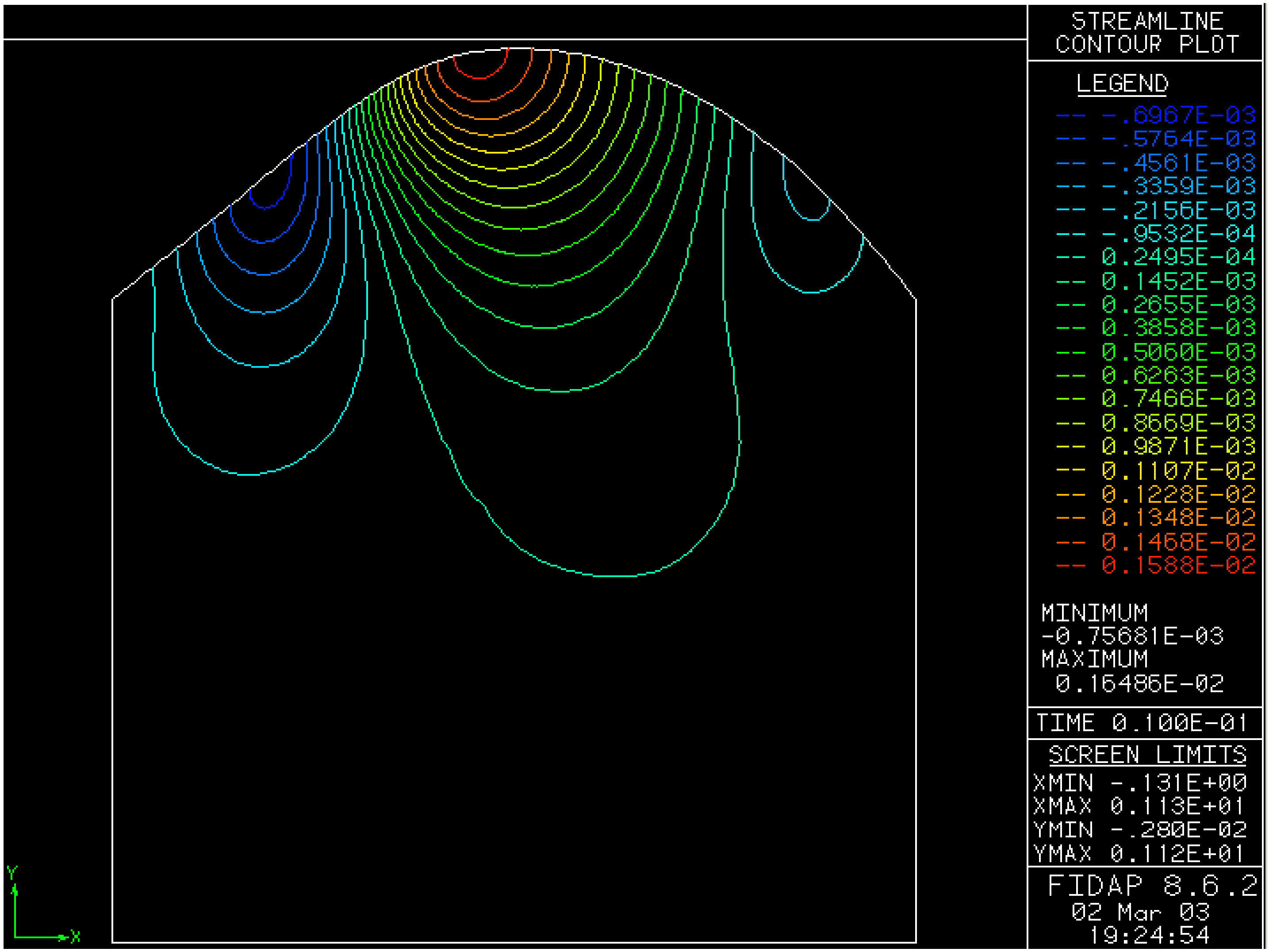

Figure 13 Stream lines at t=3sec

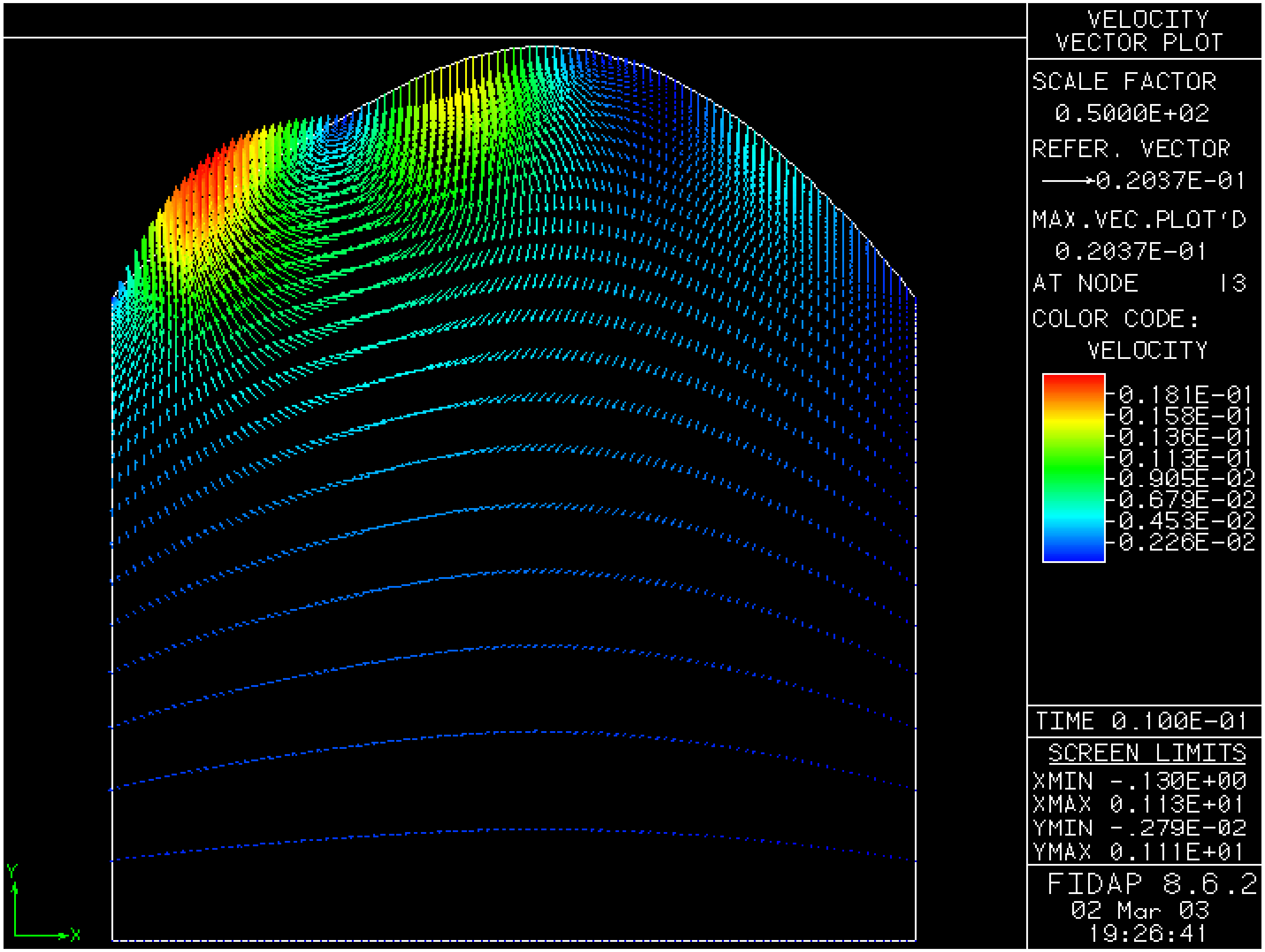

Figure 14 Velocity field at t=4sec

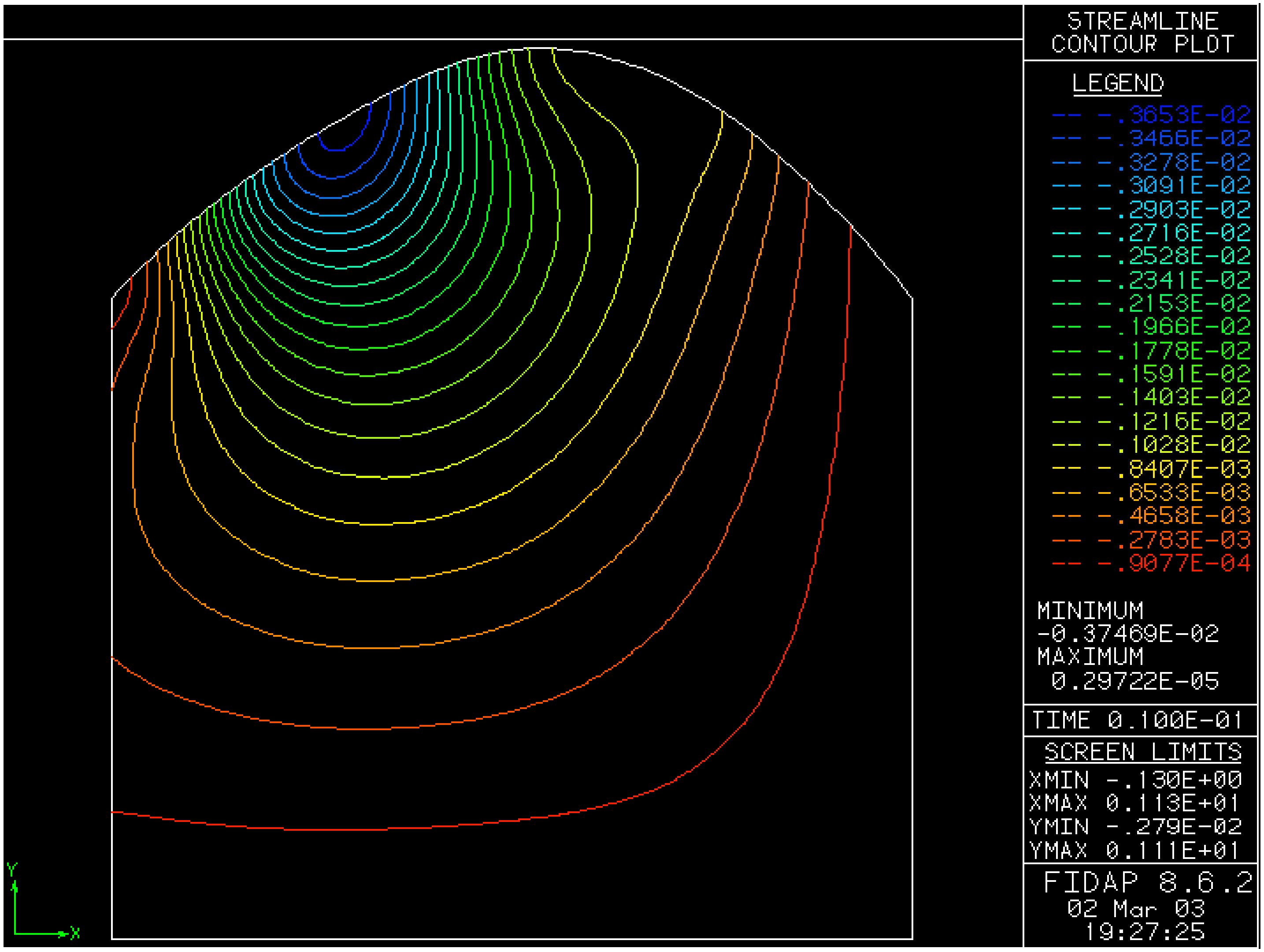

Figure 15 Stream lines at t=4sec

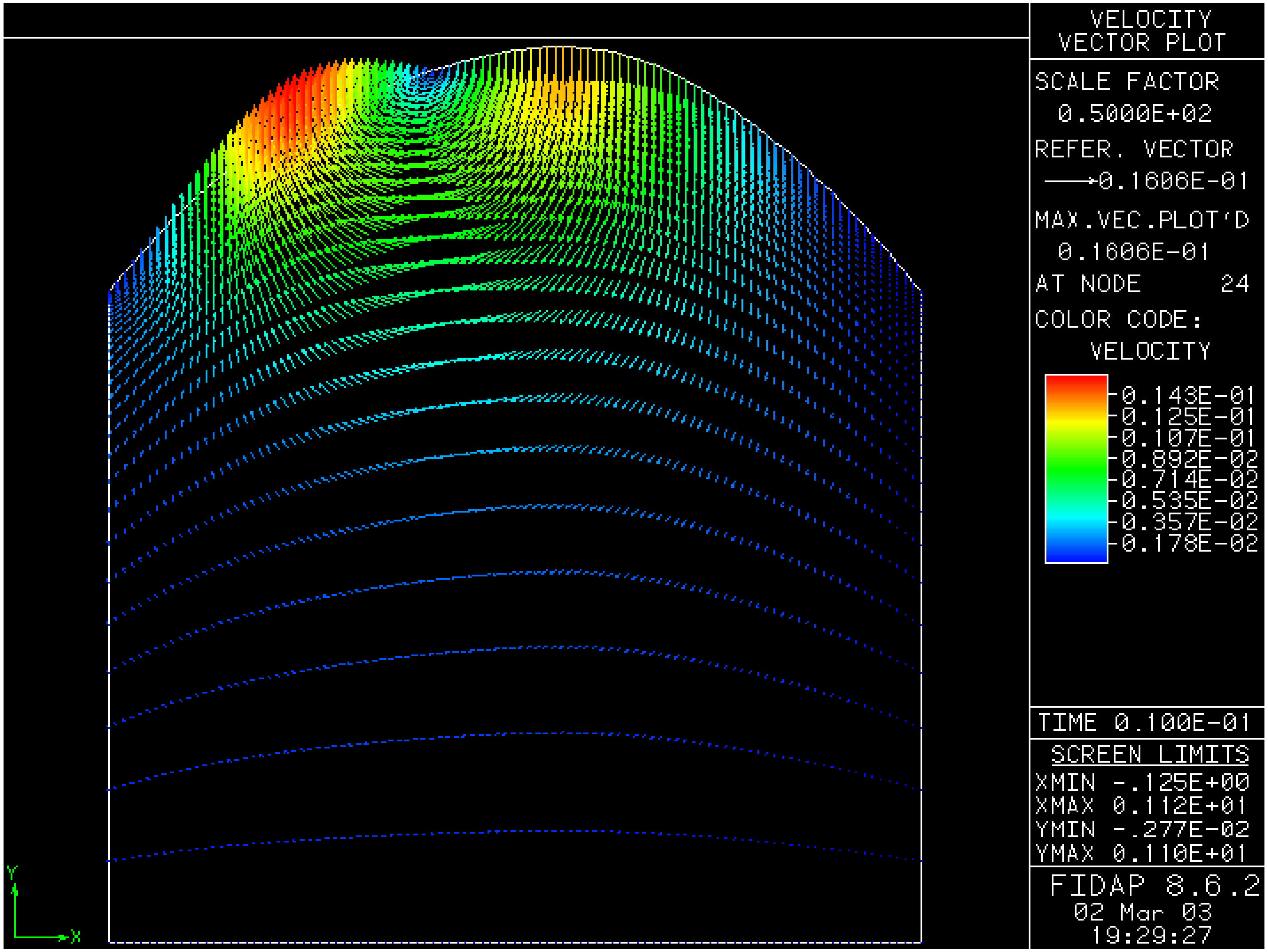

Figure 16 Velocity field at t=5sec

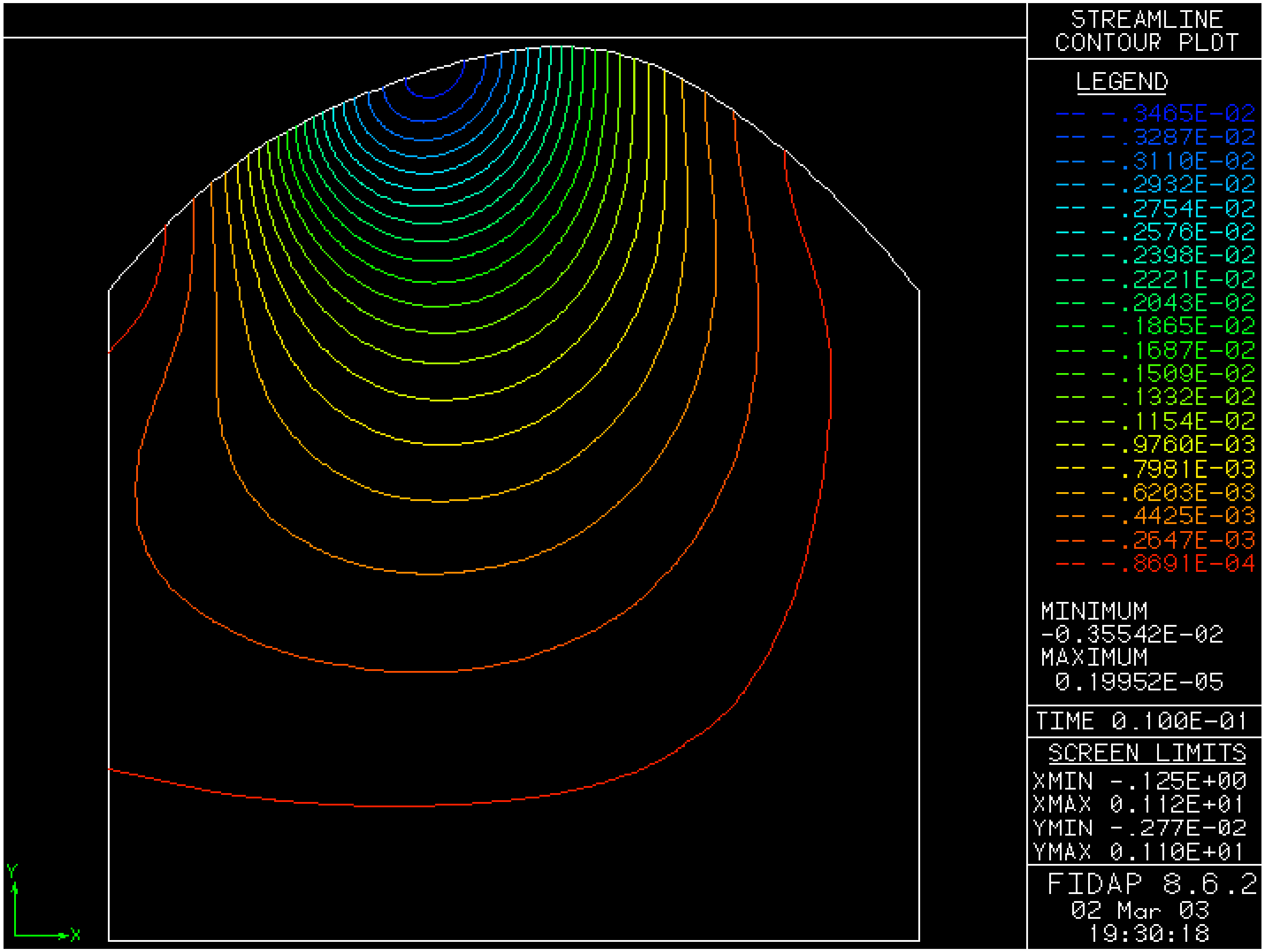

Figure 17 Stream lines at t=5sec

Future work

This work presented a numerical method to solve the elastic membrane – fluid interaction in a closed reservoir. The next necessary step is to develop the stability and convergence analysis of this method. An interesting extension to the current model will be using a more complicated elasticity or viscosity models.

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